

# Cooling through Optimal Control of Quantum Evolution

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Nonadiabatic unitary evolution with tailored time-dependent Hamiltonians can prepare systems of cold atomic gases with various desired properties. For a system of two one-dimensional quasicondensates coupled with a time-varying tunneling amplitude, we show that the optimal protocol, for maximizing any figure of merit in a given time, is bang-bang, *i.e.*, the coupling alternates between only two values through a sequence of sudden quenches. Minimizing the energy of one of the quasicondensates with such nonadiabatic protocols, and then decoupling it at the end of the process, can result in effective cooling beyond the current state of the art.

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Recent advances in the physics of ultracold atoms, brought about by ingenious cooling techniques such as evaporative and laser cooling, have stirred up great interest in the nonequilibrium dynamics of many-body quantum systems [1–5]. Despite remarkable progress, however, the quest for lower temperatures still continues. For example, creating ground states of important model Hamiltonians, such as the two-dimensional fermionic Hubbard model, has remained elusive. In addition to cooling, preparing systems with other desired characteristics, such as, *e.g.*, number-squeezed ones, is of considerable interest due to the potential applications to quantum metrology and precision measurements. Focusing on a pair of two coupled elongated (assumed one-dimensional) quasicondensates (hereafter referred to simply as condensates despite lack of true long-range order), we propose a scheme for preparing cold atomic systems with custom-ordered figures of merit through optimal control of their nonequilibrium quantum dynamics. As we will show, the large degree of dynamical control over these systems provides, among others, a new means of bringing them even closer to zero temperature.

Let us begin by giving a few examples of experimentally relevant quantities one can optimize in cold atom systems:

1. **Effective cooling:** by minimizing quantities such as the excess energy, number of quasiparticle excitations, or the trace distance between the density matrix of the system and its zero-temperature density matrix.
2. **Phase coherence:** by minimizing the fluctuations of the relative phase between two condensates (with spatially fluctuating phases), which is important for matter-wave interferometry [6–10].
3. **Number squeezing:** by minimizing the particle number fluctuations of a system [11, 12], which is important, *e.g.*, in precision measurements [13].

Focusing on effective cooling, we show in this paper that (at least) one of the condensates in the system of Fig. 1 can be cooled down by a factor of 5 with our proposed method under reasonable experimental conditions. This is not a fundamental

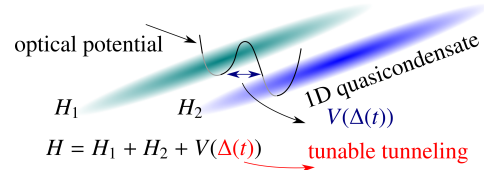


FIG. 1: Two coupled one-dimensional condensates. The tunneling amplitude  $\Delta(t)$  can be tuned by raising or lowering the potential barrier.

bound, however, and cooling by several orders of magnitude is in principle possible for highly asymmetric systems.

Let us now formulate our proposal generically. Consider a quantum system with Hamiltonian  $H$ , which is comprised of two coupled subsystems:  $H = H_1 + H_2 + V$ , where  $H_i$  is the Hamiltonian of subsystem  $i$  and  $V$  is the coupling Hamiltonian. Generically,  $V$  is a sum of certain local terms, with some coupling constants  $\{\lambda\}$ . We assume that i) we have time-dependent control over the coupling constants  $\{\lambda(t)\}$ , *i.e.*, within a range determined by the experimental constraints, we can tune them to any value as a function of time, ii) for all initial  $\{\lambda_0\}$ , we can prepare the system at inverse temperature  $\beta_0$  with the current state-of-the-art cooling methods, and, iii) we have a fixed time  $\tau$  to carry out a dynamical process (by tuning the Hamiltonian), during which the system undergoes quantum-coherent unitary evolution.

Our goal is to find an optimal protocol  $\{\lambda(t)\}$  such that, at the end of the process ( $t = \tau$ ), the energy of subsystem 1, or some other custom-ordered cost function, is minimized. For a given protocol, the density matrix evolves as  $\dot{\rho}(t) = i[H(\{\lambda(t)\}), \rho(t)]$  with initial conditions determined by the thermal state at  $t = 0$ . Thus,  $\rho(\tau)$ , and, consequently, cost functions such as  $\mathcal{E}_1(\tau) = \text{tr}[H_1 \rho(\tau)]$  are functionals of  $\{\lambda(t)\}$ ,  $0 < t < \tau$ . Notice that if we decouple the two subsystems at time  $\tau$ , the energy of subsystem 1 will remain equal to  $\mathcal{E}_1(\tau)$  for all subsequent times.

The key question addressed in this paper is how to minimize this functional of  $\{\lambda(t)\}$ . We find that i) the Pontryagin's maximum principle provides a deep understanding of the struc-

ture of such protocols, and ii) the simulated annealing method used in Ref. [14] gives a simple and generic way for performing such optimization. In simulated annealing, we discretize time, approximate an arbitrary protocol by a piece-wise constant one, and perform direct (classical) Monte-Carlo (MC) simulations with kinetic moves consisting of small random displacements of randomly chosen pieces of the protocol [14].

Let us now discuss the specific system studied in this paper, i.e., a pair of coupled one-dimensional condensates of interacting atoms with Hamiltonian  $H = H_1 + H_2 + V$ , where

$$H_i = \frac{v_i}{2} \int dx \left[ \frac{\pi}{g_i} \Pi_i^2(x) + \frac{g_i}{\pi} (\partial_x \Phi_i(x))^2 \right]. \quad (1)$$

For  $i = 1, 2$ ,  $\Pi_i(x)$  is the conjugate momentum to bosonic field  $\Phi_i(x)$ , and the coupling term has a sine-Gordon form  $V = -2\frac{\Delta}{a} \int dx \cos[\Phi_1(x) - \Phi_2(x)]$  [15]. Physically,  $\Phi_i(x)$  and  $\Pi_i(x)$  respectively represent the phase and the density fluctuations (with respect to a constant background density) of condensate  $i$  at position  $x$ , and,  $v_i$  and  $g_i$  are respectively the sound velocity and the Luttinger parameter. As seen in Fig. 1,  $\frac{\Delta}{a}$  is an effective tunneling amplitude per length ( $a$  is a microscopic length scale), which can be tuned by changing the height of the optical potential barrier.

A comment on the dimensions of the quantities above is in order. We have set  $\hbar$  to unity, and identified the units of time and inverse energy. Representing length and energy by  $\ell$  and  $\varepsilon$  respectively, the field  $\Phi(x)$  is dimensionless, its conjugate momentum  $\Pi(x)$  has dimension  $\ell^{-1}$ , and,  $v$  and  $\Delta$  respectively have dimension  $\ell\varepsilon$  and  $\varepsilon$ . Let us now use the harmonic approximation (i.e., expand the cosine term around  $\Phi_1(x) - \Phi_2(x) = 0$  and keep the leading quadratic term). This approximation is justified (at least for the initial equilibrium state of two coupled condensates) in the limit of large Luttinger parameters where the cosine term is relevant. As we will check *a posteriori*, although the differences  $\Phi_1(x) - \Phi_2(x)$  typically increase by an optimal evolution designed to cool one of the condensates, for some range of parameters, one can keep them reasonably small during the evolution so that the harmonic approximation remains valid. We can then write the Hamiltonian in momentum space as a collection of harmonic oscillators:

$$H = \sum_i \sum_{q>0} \left[ \frac{v_i \pi}{4g_i} (\Pi_q^{\Re i})^2 + \frac{v_i g_i}{\pi} q^2 (\Phi_q^{\Re i})^2 \right] + \sum_{q>0} 2\Delta (\Phi_q^{\Re 1} - \Phi_q^{\Re 2})^2 + \Re \leftrightarrow \Im, \quad (2)$$

where the superscript  $\Re$  ( $\Im$ ) indicates the real (imaginary) part. Note that  $\Phi_q$  and  $\Pi_q$  respectively have dimension  $\ell^{1/2}$  and  $\ell^{-1/2}$ . We have not included in Hamiltonian (2) the  $q = 0$  term  $H_0 = \frac{\pi}{2L} \sum_i \frac{v_i}{g_i} (N^i - N_0^i)^2 + \frac{\Delta L}{a} (\Phi_0^1 - \Phi_0^2)^2$ , which is responsible for changing the particle number  $N^i$  of condensate  $i = 1, 2$  ( $\Phi_0^i$  is conjugate to  $N^i$  and  $\frac{N_0^i}{L}$  is the background density with  $L$  representing the system size) [16]. Our goal here is to reduce the energy without reducing the particle number,

and neglecting  $H_0$  is a reasonable approximation if  $\Delta$  is not too large. Note that to prepare number-squeezed states with optimal control, we need to work only with a single-mode Hamiltonian  $H_0$  [12].

For a given protocol  $\Delta(t)$ , each mode  $q$  in the Hamiltonian (2) evolves independently. Although the modes do not interact in Eq. (2), they all evolve with the same protocol  $\Delta(t)$ , which induces correlations between them. Therefore, we have a fundamentally many-mode problem even without (subleading) mode-coupling terms, which we have neglected. The first step, however, is to analyze the dynamics of a single mode  $q$  consisting of just two coupled harmonic oscillators, namely,  $H = H_1 + H_2 + H_{12}$ , where  $H_i = \frac{1}{2}(p_i^2/m_i + k_i x_i^2) \simeq \omega_i a_i^\dagger a_i$  and  $H_{12} = \frac{\lambda}{2}(x_1 - x_2)^2$  with

$$m_i = \frac{2g_i}{\pi v_i}, \quad k_i = \frac{2}{\pi} v_i g_i q^2, \quad \lambda = \frac{4\Delta}{a}. \quad (3)$$

Let us assume the initial thermal state is prepared at  $\lambda = \lambda_0$ . We then evolve the system with a time-dependent protocol  $\lambda(t)$  (with the constraint  $0 < \lambda(t) < \lambda_{\max} = 4\Delta_{\max}/a$ ). For any  $\lambda$ , we can write the single-mode Hamiltonian as  $H = \frac{1}{2} P^T P + \frac{1}{2} X^T K(\lambda) X$ , where  $P^T = (\frac{p_1}{\sqrt{m_1}}, \frac{p_2}{\sqrt{m_2}})$ ,  $X^T = (\sqrt{m_1} x_1, \sqrt{m_2} x_2)$  and

$$K(\lambda) = \begin{pmatrix} (k_1 + \lambda)/m_1 & -\lambda/\sqrt{m_1 m_2} \\ -\lambda/\sqrt{m_1 m_2} & (k_2 + \lambda)/m_2 \end{pmatrix}.$$

We can then diagonalize the above symmetric matrix as  $K(\lambda) = Q(\lambda) \Omega(\lambda) Q^T(\lambda)$ , where  $Q(\lambda)$  is an orthonormal matrix of eigenvectors and  $\Omega = \text{diag}(\bar{\omega}_1^2, \bar{\omega}_2^2)$ , with  $\bar{\omega}_i$  a normal-mode frequency. In terms of  $\bar{\omega}_{1,2}(\lambda_0)$ , the initial density matrix is given by

$$\rho_0 = \frac{1}{Z} e^{-\beta_0 \bar{\omega}_1(\lambda_0) \bar{a}_1^\dagger(\lambda_0) \bar{a}_1(\lambda_0)} e^{-\beta_0 \bar{\omega}_2(\lambda_0) \bar{a}_2^\dagger(\lambda_0) \bar{a}_2(\lambda_0)},$$

where  $\bar{a}_j(\lambda_0)$  is the annihilation operator for normal-mode  $j = 1, 2$ , which can be written in terms of the annihilation operators  $a_i$  of oscillators  $i = 1, 2$  as

$$\bar{a}_j = \frac{1}{2} \sum_k Q_{kj} (\mathcal{F}_{jk} a_k + \mathcal{G}_{jk} a_k^\dagger), \quad (4)$$

$$\mathcal{F}_{jk} \equiv \sqrt{\frac{\bar{\omega}_j}{\omega_k}} + \sqrt{\frac{\omega_k}{\bar{\omega}_j}}, \quad \mathcal{G}_{jk} \equiv \sqrt{\frac{\bar{\omega}_j}{\omega_k}} - \sqrt{\frac{\omega_k}{\bar{\omega}_j}}. \quad (5)$$

For a system evolving with  $\lambda(t)$ , we can write the Heisenberg annihilation operator of oscillator 1 (or 2) in terms of the initial normal-mode operators as

$$a_1(t) = \sum_i [u_i(t) \bar{a}_i(\lambda_0) + v_i(t) \bar{a}_i^\dagger(\lambda_0)],$$

where  $u_i$  and  $v_i$  are some complex coefficients satisfying the following constraint:

$$|u_1|^2 + |u_2|^2 - |v_1|^2 - |v_2|^2 = 1. \quad (6)$$

The initial condition (at  $t = 0$ ) for the coefficients above are obtained by inverting Eq. (4), i.e.,  $a_j = \frac{1}{2} \sum_k Q_{jk} (\mathcal{F}_{kj} \bar{a}_k - \mathcal{G}_{kj} \bar{a}_k^\dagger)$ .

To compute  $u_i(\tau)$  and  $v_i(\tau)$ , it is convenient to consider a piece-wise constant protocol determined by a sequence  $(\lambda_i, t_i)$ , for  $i = 1 \dots n$ , so that  $a_1(\tau) = e^{iH(\lambda_1)t_1} \dots e^{iH(\lambda_n)t_n} a_1(0) e^{-iH(\lambda_n)t_n} \dots e^{-iH(\lambda_1)t_1}$ . Using the commutation relation

$$[\bar{a}_i, H(\lambda)] = \frac{\bar{\omega}_i}{2} (\bar{a}_i - \bar{a}_i^\dagger) + \frac{1}{2} \sum_j \frac{\bar{K}_{ij}(\lambda)}{\sqrt{\bar{\omega}_i \bar{\omega}_j}} (\bar{a}_j + \bar{a}_j^\dagger),$$

with  $\bar{K}(\lambda) = Q^T(\lambda_0) K(\lambda) Q(\lambda_0)$ , we then find that these coefficients at  $t = \tau$  are obtained by integrating the following equations of motion from  $t = 0$  to  $t = \tau$ :

$$\dot{u}_j = \frac{1}{2i} \left[ \sum_k (u_k - v_k) \frac{\bar{K}_{jk}(\tau - t)}{\sqrt{\bar{\omega}_j \bar{\omega}_k}} + (u_j + v_j) \bar{\omega}_j \right], \quad (7)$$

$$\dot{v}_j = \frac{1}{2i} \left[ \sum_k (u_k - v_k) \frac{\bar{K}_{jk}(\tau - t)}{\sqrt{\bar{\omega}_j \bar{\omega}_k}} - (u_j + v_j) \bar{\omega}_j \right], \quad (8)$$

where all normal-mode frequencies  $\bar{\omega}$  are calculated at  $\lambda = \lambda_0$ . Notice that the equations above depend on the final time  $\tau$ , and should not be used for computing  $u_i$  and  $v_i$  at  $t \neq \tau$ . When working at fixed  $\tau$ , it is helpful to define  $\tilde{\lambda}(t) \equiv \lambda(\tau - t)$ , which makes the equations local in time. Finding the optimal  $\tilde{\lambda}$  immediately yields the optimal  $\lambda$ .

The equations of motion (7) and (8), together with their initial conditions, uniquely determine  $a_1(\tau)$  as a functional of  $\lambda(t)$ . (Notice that the same equations with different initial conditions can be used to find  $a_2(\tau)$  as well.) Our goal is to minimize an appropriate cost function, such as the excess energy of oscillator 1, over all permissible controls  $\lambda(t)$ . For a single oscillator, the excess energy is proportional to the average number of excitations, which can be written as  $\langle n_1(t) \rangle = \text{tr} [a_1^\dagger(t) a_1(t) \rho_0]$ . In terms of the dynamical variables  $u_i(t)$  and  $v_i(t)$ , the above trace simplifies to

$$\langle n_1(t) \rangle = \sum_i |u_i(t)|^2 \bar{n}_i(0) + |v_i(t)|^2 (1 + \bar{n}_i(0)), \quad (9)$$

where  $\bar{n}_i(0) \equiv \text{tr} [\bar{a}_i^\dagger(\lambda_0) \bar{a}_i(\lambda_0) \rho_0] = (e^{\beta_0 \bar{\omega}_i(\lambda_0)} - 1)^{-1}$ .

Note that exactly the same formulation describes the many-mode problem [Eq. (2)]. In this case, we have to multiply the number of dynamical variables by the number of modes. The equations of motion [Eqs. (7) and (8)] still hold for each mode, with parameters depending on  $q$  as in Eq. (3). Appropriate many-mode cost functions can be constructed from cost functions for individual modes. For example, we can simply add  $\langle n_1^q(t) \rangle$  to obtain the total number of excitations  $\langle \mathcal{N}_1(t) \rangle$  in condensate 1, or weight them by the mode frequency to find the total excess energy  $\langle \mathcal{E}_1(t) \rangle$  in the condensate:

$$\langle \mathcal{N}_1(t) \rangle = 2 \sum_{0 < q < \Lambda} \langle n_1^q(t) \rangle, \quad \langle \mathcal{E}_1(t) \rangle = 2v_1 \sum_{0 < q < \Lambda} q \langle n_1^q(t) \rangle, \quad (10)$$

where the factor of two accounts for real and imaginary components of Hamiltonian (2) and  $\Lambda$  is a momentum cutoff. Additionally, we may also consider  $\langle C_1(t) \rangle = \sum_{0 < q < \Lambda} \langle n_1^q(t) \rangle / q$ , which is relevant for enhancing the fringe contrast of matter-wave interferometry experiments [17].

The problem we have formulated thus far is a typical problem in optimal control theory applied to quantum dynamics [14, 18–22]: we have a set of dynamical variables with given initial conditions ( $u_i$  and  $v_i$  in our case), which evolve with given equations of motion [Eqs. (7) and (8)] that depend on some admissible control parameter(s) ( $0 < \lambda(t) < \lambda_{\max}$ ). The challenge is to find an admissible optimal control such that a given cost function of the dynamical variables [Eq. (10) in our case] is minimized at a given time  $\tau$ .

Let us now turn to the main questions of this work: What do the optimal  $\lambda(t)$  protocols look like? How can we find them? How much can they cool a system? Using Pontryagin's maximum principle, we argue that optimal protocols are *bang-bang*, i.e.,  $\lambda(t)$  is either zero or equal to  $\lambda_{\max}$  at any given time. As mentioned earlier, we demonstrate that a direct simulated-annealing calculation can yield these optimal protocols. We also find that, depending on the parameters of the problem, it is possible to significantly cool down one of the condensates.

Let us now briefly review Pontryagin's maximum principle. Consider a set of dynamical variables  $\{x(t)\}$  that satisfy the equations of motion  $\dot{x}_j = f_j(\{x, \alpha\})$ , with  $x_j(0) = x_j^0$ , for a set of admissible controls  $\{\alpha(t)\}$ . The goal is to maximize a payoff function  $g(\{x(\tau)\})$  over all such  $\{\alpha(t)\}$ . The key to Pontryagin's maximum principle is the following *optimal-control Hamiltonian*:

$$\mathcal{H}(\{x, p, \alpha\}) = \sum_j p_j(t) f_j(\{x, \alpha\}), \quad (11)$$

where  $p_j(t)$  is a “momentum” conjugate to  $x_j(t)$ . The Pontryagin's theorem states that for the optimal control  $\{\alpha^*(t)\}$ , and the corresponding  $\{x^*, p^*\}$ , we have

$$\mathcal{H}^* \equiv \mathcal{H}(\{x^*, p^*, \alpha^*\}) = \max_{\{\alpha\}} \mathcal{H}(\{x^*, p^*, \alpha\}), \quad (12)$$

where  $x$  and  $p$  satisfy  $\dot{x}_j^* = \frac{\partial \mathcal{H}^*}{\partial p_j^*}$  and  $\dot{p}_j^* = -\frac{\partial \mathcal{H}^*}{\partial x_j^*}$  with boundary conditions  $x_j^*(0) = x_j^0$  and  $p_j^*(\tau) = \frac{\partial}{\partial x_j^*} g(\{x^*(\tau)\})$ . It is now easy to observe that since, for all modes  $q$ , Eqs. (7) and (8) are linear in  $\lambda(t)$ , the optimal-control Hamiltonian [Eq. (11)] is also a linear function of  $\lambda(t)$  in our case. We then immediately deduce from Eq. (12) that, unless  $\mathcal{H}^*$  identically vanishes over a finite time interval, the control  $\lambda(t)$  can only take two values, namely, zero and  $\lambda_{\max}$ .

The Pontryagin equations are not easy to solve numerically for many modes. We thus use our direct MC method, without utilizing any assumptions regarding the bang-bang nature of the protocol. The kinetics of the simulated annealing consist of varying a randomly chosen  $\lambda_i$  (of the discretized protocol) by a small random amount. As found in Ref. [14], such simulations converge very well in the number of discretization points. To compute the cost function (in each

MC step), it is convenient to define new dynamical variables  $\phi_j \equiv \sqrt{\bar{\omega}_j(\lambda_0)}(u_j + v_j)$  and  $\theta_j \equiv (u_j - v_j)/\sqrt{\bar{\omega}_j(\lambda_0)}$ , which satisfy  $|\dot{\phi}\rangle = -i\bar{K}(\tilde{\lambda})|\theta\rangle$  and  $|\dot{\theta}\rangle = -i|\phi\rangle$  in matrix notation (this change of variables allows us to diagonalize  $2 \times 2$  matrices instead of  $4 \times 4$  ones). By solving the above equations in terms of the eigenvalues and eigenvectors of the  $2 \times 2$  matrix  $\bar{K}(\tilde{\lambda}_m) = \bar{K}(\lambda_{n-m+1})$ , we can then write simple recursion relation for  $|\phi\rangle$  and  $|\theta\rangle$ , which yield their values at time  $\tau$  after  $n$  iterations.

One comment is in order before proceeding. In addition to optimizing over  $\lambda(t)$ , we have the freedom to choose the initial  $\lambda_0$  anywhere between zero and  $\lambda_{\max}$ . There is a rigorous lower bound on  $\langle n_1 \rangle$  [Eq. (9)], which follows from constraint (6):  $\langle n_1(t) \rangle \geq n_{\min} \equiv \min(\langle \bar{n}_1(0) \rangle, \langle \bar{n}_2(0) \rangle)$ . We can show that  $n_{\min}$  is a decreasing function of  $\lambda_0$ . This suggests that it may be advantageous to set  $\lambda_0 = \lambda_{\max}$ . Although the actual cost functions we are able to reach by our minimization procedure are typically much larger than this lower bound, by trying several values of  $\lambda_0$  in our numerics, we have found that the best cooling is in fact achieved for  $\lambda_0 = \lambda_{\max}$ . In Fig. 2a, we show a typical protocol obtained by MC simulations. We converge to a bang-bang protocol by an unbiased simulation, which samples all the intermediate values of  $\lambda$ , and, *a priori*, does not assume anything about the shape of the protocol. Surprisingly, minimizing  $N_1$ ,  $\mathcal{E}_1$ , or  $C_1$  leads to very similar, albeit nonidentical, protocols (here we show the protocol obtained by minimizing  $N_1$ ). To further check the consistency with the Pontryagin's theorem, we also computed the derivative  $\partial_\lambda \mathcal{H}$ , the sign of which determines  $\lambda(t)$  through Eq. (12), for cost function  $\langle N_1(\tau) \rangle$ .

To construct  $\mathcal{H}$ , we need to treat the real and imaginary parts of  $\phi_j$  and  $\theta_j$  as separate dynamical variables with their own conjugate momenta. We can then construct a complex variable  $\pi_j^\phi$ , whose real (imaginary) part is the conjugate momentum to the real (imaginary) part of  $\phi_j$ , and similarly for  $\theta_j$ . For each  $q$ , we then have  $|\dot{\pi}^\phi\rangle = -i|\pi^\theta\rangle$  and  $|\dot{\pi}^\theta\rangle = -i\bar{K}(\tilde{\lambda})|\pi^\phi\rangle$ . The boundary conditions at  $t = \tau$  depend on the cost function [see the boundary conditions below Eq. (12)] and, for  $\langle N_1(\tau) \rangle$ , can be written as  $\pi_i^\phi(\tau) = \theta_i(\tau) - (2\langle \bar{n}_i(0) \rangle + 1) \frac{1}{\bar{\omega}_i(0)}$  and  $\pi_i^\theta(\tau) = \phi_i(\tau) - (2\langle \bar{n}_i(0) \rangle + 1) \bar{\omega}_i(0) \theta_i(\tau)$ . Given a protocol  $\lambda(t)$ , we can solve for  $\phi$  and  $\theta$  forward in time, construct  $\pi^\phi(\tau)$  and  $\pi^\theta(\tau)$  from the boundary conditions above, solve for  $\pi^\phi$  and  $\pi^\theta$  backward in time and finally construct  $\partial_\lambda \mathcal{H} = \sum_q \langle \pi_q^\phi | \partial_\lambda \bar{K}_q(\tilde{\lambda}) | \theta_q \rangle$ , which immediately yields  $\partial_\lambda \mathcal{H}$ . The results are shown in Fig. 2a, and show excellent agreement with the simulations.

In Fig. 2b, we show how the cost function  $\langle N_1(t) \rangle$  changes when evolving with the optimal protocol. An interesting feature of the evolution is that  $\frac{d\langle N_1(t) \rangle}{dt} \neq 0$  just before quenching to  $\lambda(t) = 0$ . Keeping the subsystems coupled would do a better job in reducing  $\langle N_1(t) \rangle$  locally (in time) but would not lead to global optimization in total time  $\tau$ . We can also check the harmonic approximation *a posteriori* by computing  $\frac{1}{L} \int dx [\Phi_1(x) - \Phi_2(x)]^2$ . We find that as long as the approximation is valid initially, and  $\lambda_{\max}$  is large enough, this quan-

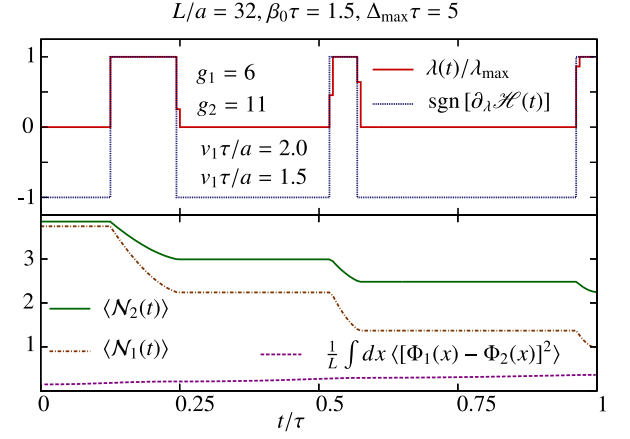


FIG. 2: a) Typical protocols obtained with unbiased simulated annealing, and the derivative of the optimal control Hamiltonian with respect to control  $\lambda$ , whose sign determines the protocol. The simulations converge to bang-bang protocols in excellent agreement with  $\text{sgn}[\partial_\lambda \mathcal{H}]$ . b) Reduction of  $\langle N_i(t) \rangle$  due to evolution with the two optimal protocols above. Cooling condensate 1, may also cool down the other condensate for free. For large Luttinger parameters, the argument of the cosine term remains much smaller than one during the evolution, i.e., the harmonic approximation remains valid.

tity remains smaller than one and the harmonic approximation holds throughout the evolution (if the system is not too long, the spatial fluctuations of  $[\Phi_1(x) - \Phi_2(x)]^2$  are small). Interestingly, the optimal protocol designed for reducing the energy of condensates 1 turns out to also cool down condensate 2. This is not a violation of the second law of thermodynamics as our process is not cyclic: we start from two coupled condensates with  $H = H_1 + H_2 + V$ , and end up with two decoupled ones with  $H = H_1 + H_2$ . The process only reduces the expectation value  $\langle H_1 + H_2 \rangle$ , while  $\langle H_1 + H_2 + V \rangle$ , which corresponds to the initial Hamiltonian, actually increases.

The effective cooling described here is an out-of-equilibrium reduction of the excess energy, and does not imply thermal equilibrium. If the low-energy system equilibrates afterwards, however, it will have a lower temperature. To directly bring the system close to thermal equilibrium, one can instead minimize the trace distance between the density matrix and thermal density matrices at varying target temperatures, and find a balance between a small trace distance and a low target temperature. We do not pursue this approach here because, under realistic circumstances, each decoupled condensate is expected to eventually decohere, and reach an effective temperature determined by its excess energy [23].

Finally, we discuss the performance of our optimal protocols, which depends on several (dimensionless) parameters, including the two Luttinger parameters,  $\beta_0\tau$ ,  $v_1\tau/a$ , and  $L/a$ . However,  $\Delta_{\max}\tau$  and the ratio  $v_2/v_1$  seem to have the most pronounced effect on the performance (measured by the ratio of the achieved energy to equilibrium energy at  $\beta_0$ ). Typically, the energy can be reduced by a factor of 3 to 5 when  $v_2/v_1$  is of order unity. For a highly asymmetric system with



$v_2/v_1 = 100$ , we achieved an energy reduction by a factor of 40 with a system size of  $L/a = 64$  and other dimensionless parameters of order unity.

Let us now comment on a possible extension of our scheme to arbitrary systems. To perform our MC simulations, we need to be able to efficiently compute desired cost functions for any allowed protocol (which is the case for our system in the harmonic approximation). Cooling down more complicated systems such as the two-dimensional fermionic Hubbard model (or even our system in regimes where the full sine-Gordon term is needed) is of considerable interest for quantum simulations. The generality of our MC method, however, raises an intriguing possibility for a *universal* approach: if one can automate the processes of system initialization (initial cooling), unitary evolution (with a tailored protocol), and measurement of the figure of merit (e.g., energy), then the *system itself* can be used to perform such MC simulations. The cost function can be measured (instead of computed), and then fed into the MC algorithm. This would provide a powerful means of preparing desired states in arbitrary systems, and may open the door to the quantum simulation of unsolved condensed-matter models. Such integration of experiment and simulation has in fact been applied to the control of chemical reactions [24], and more recently to some aspects of cold atom experiments [25].

In summary, we demonstrated that nonadiabatic optimal control of quantum evolution can be used to push the boundaries of atomic cooling. Contrary to the conventional association of nonadiabatic effects with heating, we showed that breaking away from the adiabatic limit, in a controlled way, can in fact help cool down quantum systems. We applied this idea to a system of two coupled elongated condensates. Through simple and direct MC simulations, we found optimal protocols which agree with theoretical predictions based on Pontryagin's maximum principle, and are effective in reducing the excess energy. Such MC simulations can be potentially performed by the system itself giving access to a universal cooling scheme.

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- [2] T. Kinoshita, T. Wenger, and D. S. Weiss, *Nature* **440**, 900 (2006).
- [3] L. E. Sadler, J. M. Higbie, S. R. Leslie, M. Vengalattore, and D. M. Stamper-Kurn, *Nature* **443**, 312 (2006).
- [4] S. Hofferberth, I. Lesanovsky, B. Fischer, T. Schumm, and J. Schmiedmayer, *Nature* **449**, 324 (2007).
- [5] I. Bloch, J. Dalibard, and W. Zwerger, *Rev. Mod. Phys.* **80**, 885 (2008).
- [6] M. R. Andrews, C. G. Townsend, H.-J. Miesner, D. S. Durfee, D. M. Kurn, and W. Ketterle, *Science* **275**, 637 (1997).
- [7] A. Polkovnikov, E. Altman, and E. Demler, *Proc. Natl. Acad. Sci.* **103**, 6125 (2006).
- [8] V. Gritsev, E. Altman, E. Demler, and A. Polkovnikov, *Nat. Phys.* **2**, 705 (2006).
- [9] G.-B. Jo, J.-H. Choi, C. A. Christensen, Y.-R. Lee, T. A. Pasquini, W. Ketterle, and D. E. Pritchard, *Phys. Rev. Lett.* **99**, 240406 (2007).
- [10] S. Hofferberth, I. Lesanovsky, T. Schumm, A. Imambekov, V. Gritsev, E. Demler, and J. Schmiedmayer, *Nat. Phys.* **4**, 489 (2008).
- [11] V. Giovannetti, S. Lloyd, and L. Maccone, *Nature* **455**, 1216 (2008).
- [12] J. Grond, J. Schmiedmayer, and U. Hohenester, *Phys. Rev. A* **79**, 021603 (2009).
- [13] V. Giovannetti, S. Lloyd, and L. Maccone, *Science* **306**, 1330 (2004).
- [14] A. Rahmani and C. Chamon, *Phys. Rev. Lett.* **107**, 016402 (2011).
- [15] V. Gritsev, A. Polkovnikov, and E. Demler, *Phys. Rev. B* **75**, 174511 (2007).
- [16] F. D. M. Haldane, *Phys. Rev. Lett.* **47**, 1840 (1981).
- [17] T. Kitagawa, S. Pielawa, A. Imambekov, J. Schmiedmayer, V. Gritsev, and E. Demler, *Phys. Rev. Lett.* **104**, 255302 (2010).
- [18] U. Hohenester, P. K. Rekdal, A. Borzi, and J. Schmiedmayer, *Phys. Rev. A* **75**, 023602 (2007).
- [19] X. Chen, A. Ruschhaupt, S. Schmidt, A. del Campo, D. Guéry-Odelin, and J. G. Muga, *Phys. Rev. Lett.* **104**, 063002 (2010).
- [20] P. Doria, T. Calarco, and S. Montangero, *Phys. Rev. Lett.* **106**, 190501 (2011).
- [21] K. H. Hoffmann, P. Salamon, Y. Rezek, and R. Kosloff, *Eur. Phys. Lett.* **96**, 60015 (2011).
- [22] T. Caneva, T. Calarco, R. Fazio, G. E. Santoro, and S. Montangero, *Phys. Rev. A* **84**, 012312 (2011).
- [23] A. A. Burkov, M. D. Lukin, and E. Demler, *Phys. Rev. Lett.* **98**, 200404 (2007).
- [24] A. Assion, T. Baumert, M. Bergt, T. Brixner, B. Kiefer, V. Seyfried, M. Strehle, and G. Gerber, *Science* **282**, 919 (1998).
- [25] W. Rohringer, R. Bcker, S. Manz, T. Betz, C. Koller, M. Gbel, A. Perrin, J. Schmiedmayer, and T. Schumm, *Appl. Phys. Lett.* **93**, 264101 (2008).

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[1] M. Greiner, O. Mandel, T. Hänsch, and I. Bloch, *Nature* **419**, 51 (2002).